Homework for MDA

Due January 2, 2016

More removals, but there's still plenty to work on.

1 Roots and Quadratics

Just one left!

1. $$\int \frac{x-3}{\left(4-2x-x^2\right)^2} dx$$

2 Definite Integrals

2.1 Explicit (Mostly Improper) Integrals

1. $$\int_1^\infty \frac{dx}{x^2}$$

2. The above integral really represents a limit; write that limit explicitly.

3. $$\int_1^\infty \frac{dx}{x}$$

4. Determine the set of values of p and a for which the integral
$$\int_a^\infty \frac{dx}{x^p}$$
is convergent.

5. $$\int_{-\infty}^0 \frac{dx}{\sqrt{3-x}}$$

6. $$\int_{-\infty}^\infty xe^{-x^2} dx$$

7.
$$\int_{-2}^{\infty} \sin x \, dx$$

8.
$$\int_{0}^{3} \frac{dx}{\sqrt{3-x}}$$

9.
$$\int_{-2}^{3} \frac{dx}{x^3}$$

10.
$$\int_{0}^{\infty} \frac{dx}{x^2}$$

3 Stuff to Think About with Integrals

3.1 Volume

Find a function of x that would create a half-circle of radius r (easiest if centered on the origin). Note that a sphere can be imagined to be made up of infinitely small disks with radii given by this function and "height" dx. Find an integral that will add the volumes of all these microcylinders to get the volume of a sphere.

Similar "solids of revolution" can be created from other functions, and while they are rarely useful, practice with them can actually improve your conceptual understanding of integral calculus. Try getting the volumes of the following solids of revolution:

1. $x^2 - 4x + 5$, $1 \le x \le 4$, rotated about the x-axis
2. The region bounded by $\sqrt[3]{x}$ and $x/4$, rotated about the y-axis
3. The region bounded by $y = x^2 - 2x$ and $y = x$ rotated about $y = 4$
4. The region bounded by $2\sqrt{x-1}$ and $x - 1$ rotated about $x = -1$

3.2 Surface Area

Combining the previous two subsections, you could have a handy method for finding the surface area of a solid of revolution. Go ahead and try finding the surface area of a sphere that way.

Then try your hand at the surface areas of the following:

1. $\sqrt{9-x^2}$, $-2 \le x \le 2$ rotated about the x-axis
2. $\sqrt[3]{x}$, $1 \le y \le 2$ rotated about the y-axis

If you saw some of this in school (ahem, Dasha), try ignoring what you were told to memorize and thinking in terms of fundamentals, as I try to get you to do here. You really get a better feel for integrals after having done this.

4 Parametric Curves

1. Find the tangent line:
$$x = t^5 - 4t^3;\; y = t^2;\; \text{at } (0, 4).$$
2. Find the coordinates at which the following curve has horizontal or vertical tangents:
$$x = t^3 - 3t;\; y = 3t^2 - 9.$$
3. Find the second derivative (d^2y/dx^2) of the curve
$$x = t^5 - 4t^3;\; y = t^2.$$
4. Determine the values of t for which the following curve is concave up/down:
$$x = 1 - t^2;\; y = t^7 + t^5.$$
5. Determine the area under the following curve:
$$x = 6(\theta - \sin\theta);\; y = 6(1 - \cos\theta);\; 0 \leq \theta \leq 2\pi.$$
6. Determine the length of the following curve:
$$x = 3\sin t;\; y = 3\cos t;\; 0 \leq t \leq 2\pi.$$
7. Same for the following:
$$x = 3\sin(3t);\; y = 3\cos(3t);\; 0 \leq t \leq 2\pi.$$
8. Determine the surface area of the solid obtained by rotating the following curve around the x-axis:
$$x = \cos^3\theta;\; y = \sin^3\theta;\; 0 \leq \theta \leq \frac{\pi}{2}.$$

5 Properties of Inverse Matrices

Suppose A and B are invertible matrices of the same size and c is a nonzero scalar.

1. Find $(AB)^{-1}$.
2. Is A^{-1} necessarily invertible?
3. Find $(A^{-1})^{-1}$.
4. Find $(A^n)^{-1}$.
5. Find $(A^{-1})^n$.
6. Find $(cA)^{-1}$.
7. Find $(A^T)^{-1}$.
8. Find $(A^{-1})^T$.

6 Special Matrices

6.1 Diagonal Matrices

A *diagonal matrix* D only has nonzero elements along its main diagonal. Other ways to say this (think about these) are that it has elements d_{ii} (we might as well just call them d_i) and all other elements are 0; or that its elements are $d_i\delta_{ij}$.

Find D^{-1}. Find D^k.

6.2 Symmetric Matrices

If a matrix S is symmetric, then $S^T = S$.

Suppose A and B are symmetric, and c is a scalar. What can you say about $A \pm B$? What about cA? What about A^T? What about AB? What about A^{-1}?

Suppose C is any matrix at all—not necessarily square. What can we say about CC^T? What about C^TC?

7 LU Decomposition

This is the first way we will learn to factor a matrix; for a square matrix A, we will seek a factorization $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix.

Find an LU decomposition for the following matrices:

1.
$$\begin{bmatrix} 3 & 6 & -9 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{bmatrix}$$

2.
$$\begin{bmatrix} 2 & 3 & -4 \\ 5 & 4 & 4 \\ -1 & 7 & 0 \end{bmatrix}$$

8 A Better Approach to Systems

So far, we have been using “augmented matrices” as an approach to systems. This is far better than the school-level “substitution method” when many variables are involved, but it’s not really “natural” to linear algebra—which, as we will find, is a significant drawback if we want to be really effective at this.

Fortunately, the fix is easy. Suppose, familiarly, that we have the system

$$\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = b_2, \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = b_n.
\end{gathered}$$

Instead of writing it as an augmented matrix, we can write it as a regular matrix multiplying a vector of the variables we're solving for, equalling a vector of the right-hand-side variables. Confusing? Here it is without a human language in the way:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Thus, in linear algebra, the entire system is a single equation: $A\mathbf{x} = \mathbf{b}$. And this equation is very easy to solve for $\mathbf{x}$: $\mathbf{x} = A^{-1}\mathbf{b}$.

With that in mind, solve the following system:

$$\begin{aligned}
3x_1 + x_2 &= 6, \\
2x_2 - x_1 + 2x_3 &= -7, \\
5x_1 - x_3 &= 10.
\end{aligned}$$

You'll find that it's a bit of a pain because you have to invert a matrix. Here we can use LU decomposition to make life a bit easier. We factor the matrix $A = LU$, so now

$$A\mathbf{x} = \mathbf{b} \Rightarrow LU\mathbf{x} = \mathbf{b}.$$

Now, we do a variable substitution, substituting $\mathbf{y} \equiv U\mathbf{x}$. Now $L\mathbf{y} = \mathbf{b}$, which you'll find really easy to solve for $\mathbf{y}$. After that, it will also be easy to solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$. This is *extremely* useful when programming.

Make sure this all make sense! Then try solving the following systems:

- $$\begin{aligned}
3x_1 + 6x_2 - 9x_3 &= 0, \\
2x_1 + 5x_2 - 3x_3 &= -4, \\
x_2 - 4x_1 + 10x_3 &= 3.
\end{aligned}$$

- $$\begin{aligned}
4x_2 - 2x_1 - 3x_3 &= -1, \\
3x_1 - 2x_2 + x_3 &= 17, \\
3x_3 - 4x_2 &= -9.
\end{aligned}$$

Solve the following two systems of equations in such a way that you do very little work on the second one:

- $$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 12, \\ 2x_1 - x_2 - 2x_3 &= -1, \\ 5x_1 - 2x_2 - 3x_3 &= 3. \end{aligned}$$

- $$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 0, \\ 2x_1 - x_2 - 2x_3 &= -5, \\ 5x_1 - 2x_2 - 3x_3 &= -8. \end{aligned}$$

Once you have a feel for that, determine what conditions, if any, b_i must satisfy for the following systems to be consistent:

- $$\begin{aligned} x_1 - 2x_2 + 6x_3 &= b_1, \\ x_2 - x_1 - x_3 &= b_2, \\ x_2 - 3x_1 + 8x_3 &= b_3. \end{aligned}$$

- $$\begin{aligned} x_1 + 3x_2 - 2x_3 &= b_1, \\ 3x_3 - x_1 - 5x_2 &= b_2, \\ 2x_1 - 8x_2 + 3x_3 &= b_3. \end{aligned}$$

9 Introduction to Markov Chains

Suppose that if it's sunny one day, the probability of it being sunny the next day is 0.9; if it's rainy one day, the probability of it being sunny the next day is 0.6. What is the probability that on a completely random day it will be sunny?

10 Fun with Limits

Just have some fun and explore and do not stress.

$$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\ldots+\frac{n-1}{n^2}\right).$$

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{n+n}\right).$$

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{n}{n^2+n^2}\right).$$

$$\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\ldots+\sin\frac{(n-1)\pi}{n}\right).$$

$$\lim_{n\to\infty}\frac{1^p+2^p+\ldots+n^p}{n^{p+1}}\qquad(p>0).$$

$$\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\ldots+\sqrt{1+\frac{n}{n}}\right),$$